Homework 5

§ 15.6

Q4: Area of the region $= \frac{1}{2} \times 3 \times 3 = \frac{9}{2}$

Centroid

$$= \left(\frac{2}{9} \int_\Delta x\,dA \,, \frac{2}{9} \int_\Delta y\,dA \right)$$

Now, $\int_\Delta x\,dA = \int_0^3 \int_0^{3-x} x\,dy\,dx$

$$= \int_0^3 3x - x^2\,dx$$

$$= \frac{27}{2} - 9 = \frac{9}{2}$$

By symmetry, $\int_\Delta y\,dA = \frac{9}{2}$

Thus centroid $= (1, 1)$ (which is just the of three vertices of the triangular region!)

Q5. Area $= \frac{\pi a^2}{4}$

$$\bar{x} = \frac{4}{\pi a^2} \int_0^{\pi/2} \int_0^a r\cos\theta \, r\,dr\,d\theta$$

$$= \frac{4}{\pi a^2} \int_0^{\pi/2} \cos\theta\,d\theta \cdot \int_0^a r^2\,dr$$

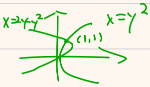
$$= \frac{4}{\pi a^2} \cdot 1 \cdot \frac{a^3}{3}$$

$$= \frac{4a}{3\pi}$$

By symmetry, $\bar{y} = \frac{4a}{3\pi}$

$\Rightarrow$ Centroid $= \left(\frac{4a}{3\pi}, \frac{4a}{3\pi} \right)$

Q14: Center of mass:



$$\int_{Region} \delta\, dA = \int_0^1 \int_{y^2}^{2y-y^2} y+1 \, dx\, dy$$

$$= \int_0^1 (y+1)(2y-2y^2)\, dy$$

$$= \int_0^1 2y-2y^3\, dy$$

$$= 1 - \frac{2}{4}$$

$$= \frac{1}{2}$$

$$\int_{Region} x\delta\, dA = \int_0^1 \int_{y^2}^{2y-y^2} (y+1)x \, dx\, dy$$

$$= \int_0^1 \frac{1}{2}(y+1)\left[(2y-y^2)^2 - (y^2)^2\right] dy$$

$$= \int_0^1 2y^2 - 2y^4 \, dy$$

$$= \frac{4}{15}$$

$$\int_{Region} y\delta\, dA = \int_0^1 \int_{y^2}^{2y-y^2} (y+1)y \, dx\, dy$$

$$= \int_0^1 2y^2 - 2y^4$$

$$= \frac{4}{15}$$

$$\Rightarrow (\bar{x}, \bar{y}) = 2\left(\frac{4}{15}, \frac{4}{15}\right) = \left(\frac{8}{15}, \frac{8}{15}\right)$$

Moment of inertia: $\int_{Region} y^2 \delta\, dA$

$$= \int_0^1 \int_{y^2}^{2y-y^2} y^2(1+y)\, dx\, dy$$

$$= \int_0^1 2y^3 - 2y^5 \, dy$$

$$= \frac{1}{6}$$

Q17: Center of mass:

$$\int \delta = \int_{-1}^{1} \int_{0}^{x^2} 7y+1 \, dy dx$$
$$= \int_{-1}^{1} \frac{7}{2} x^4 + x^2 dx$$
$$= \frac{31}{15}$$

$$\int \delta x = \int_{-1}^{1} \int_{0}^{x^2} x(7y+1) \, dy dx$$
$$= \int_{-1}^{1} \frac{7}{2} x^5 + x^3 dx$$
$$= 0$$

$$\int \delta y = \int_{-1}^{1} \int_{0}^{x^2} y(7y+1) \, dy dx$$
$$= \int_{-1}^{1} \frac{7}{3} x^6 + \frac{1}{2} x^4 dx$$
$$= \frac{13}{15}$$

$$\Rightarrow (\bar{x}, \bar{y}) = \frac{15}{31}\left(0, \frac{13}{15}\right) = \left(0, \frac{13}{31}\right)$$

Moment of Inertia $= \int \delta x^2$
$$= \int_{-1}^{1} \int_{0}^{x^2} (7y+1) x^2 \, dy dx$$
$$= \int_{-1}^{1} \frac{7}{2} x^6 + x^4 \, dx$$
$$= \frac{7}{5}$$

Q2b: By symmetry, center of mass $= \left(\frac{-1+1}{2}, \frac{3+5}{2}, \frac{-1+1}{2}\right) = (0,4,0)$

$$
\begin{aligned}
I_x &= \int_{-1}^{1}\int_{3}^{5}\int_{-1}^{1} y^2+z^2 \, dx\,dy\,dz \\
&= 2\int_{-1}^{1}\int_{3}^{5} y^2+z^2 \, dy\,dz \\
&= 2\int_{-1}^{1} \frac{98}{3} + 2z^2 \, dz \\
&= 2\left[\frac{196}{3} + \frac{4}{3}\right] = \frac{400}{3}
\end{aligned}
$$

, and $I_z = \frac{400}{3}$ by symmetry.

$$
\begin{aligned}
I_y &= \int_{-1}^{1}\int_{3}^{5}\int_{-1}^{1} x^2+z^2 \, dx\,dy\,dz \\
&= \int_{-1}^{1}\int_{3}^{5} \frac{2}{3} + 2z^2 \, dy\,dz \\
&= \int_{-1}^{1} \frac{4}{3} + 4z^2 \, dz \\
&= \frac{8}{3} + \frac{8}{3} = \frac{16}{3}
\end{aligned}
$$

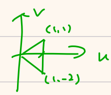
§ 15.8

Q1: a) $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} u \\ v \end{pmatrix}$

$$= \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

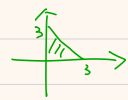
$$\Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \det\left(\frac{1}{3} \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} \right) = \frac{1}{9} \cdot (3) = \frac{1}{3}$$

b) Region: $\begin{cases} x \leq 1 \\ -2x \leq y \leq x \end{cases}$



is mapped to $\begin{cases} \frac{1}{3}(u+v) \leq 1 \\ \frac{-2}{3}(u+v) \leq \frac{1}{3}(-2u+v) \leq \frac{1}{3}(u+v) \end{cases}$

i.e. $\begin{cases} u+v \leq 3 \\ v \geq 0 \\ u \geq 0 \end{cases}$



Q6. The region is bounded by:

$2x+y=4$, $2x+y=7$, $x-y=2$, $x-y=-1$

i.e. $v=4$, $v=7$, $u=2$, $u=-1$

Moreover, $2x^2-xy-y^2 = (2x+y)(x-y)$

$= vu$

$$\Rightarrow \iint_R 2x^2 - xy - y^2 \, dx\,dy$$

$$= \int_{-1}^{2} \int_{4}^{7} vu \, \frac{1}{3} \, du\,dv$$

$$= \frac{3}{2} \cdot \frac{33}{2} \cdot \frac{1}{3} = \frac{33}{4}$$

Q18: a) $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \cos v & -u\sin v & 0 \\ \sin v & u\cos v & 0 \\ 0 & 0 & 1 \end{vmatrix} = u(\cos^2 v + \sin^2 v) = u$

b) $\dfrac{\partial(x,y,\partial z)}{\partial(u,v,w)} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \frac{1}{2} \end{vmatrix} = 3$

Q19: $x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\theta$

$\Rightarrow \dfrac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix}$

$= \cos\phi\left(\rho^2\cos\phi\sin\phi\cos^2\theta + \rho^2\cos\phi\sin\phi\sin^2\theta\right) + \rho\sin\phi\left(\rho\sin^2\phi\cos^2\theta + \rho\sin^2\phi\sin^2\theta\right)$

$= \rho^2\cos^2\phi\sin\phi + \rho^2\sin^3\phi = \rho^2\sin\phi$

Q25: It is given that

$$\frac{\int_{\substack{u^2+v^2+w^2\le 1 \\ w\ge 0}} dudvdw}{\int_{\substack{u^2+v^2+w^2\le 1 \\ w\ge 0}} dudvdw} = \left(0, 0, \frac{3}{8}\right)$$

Now, Let $u = \frac{x}{|a|}$, $v = \frac{y}{|b|}$, $w = \frac{z}{|c|}$

then $\begin{cases} \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1 \\ z \ge 0 \end{cases}$

is transformed into $\begin{cases} u^2+v^2+w^2 \le 1 \\ w \ge 0 \end{cases}$, and $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = |abc|$

And the centroid is $$\frac{\int_{\substack{u^2+v^2+w^2\le 1 \\ w\ge 0}} (|a|u, |b|v, |c|w)|abc|\, dudvdw}{\int_{\substack{u^2+v^2+w^2\le 1 \\ w\ge 0}} |abc|\, dudvdw}$$

$$= \left(0, 0, \frac{3|c|}{8}\right)$$